Written Exam for the M.Sc. in Economics 2010 (Fall Term)

Financial Econometrics A: Volatility Modelling

Final Exam: Masters course

Exam date: **10/1-2011**

3-hour open book exam.

Notes on Exam: Please note that there are a total of 9 questions which should all be answered. These are divided into Question 1 (Question 1.1-1.5) and Question 2 (Question 2.1-2.4).

Please note that the language used in your exam paper must correspond to the language of the title for which you registered during exam registration. I.e. if you registered for the English title of the course, you must write your exam paper in English. Likewise, if you registered for the Danish title of the course or if you registered for the English title which was followed by "eksamen på dansk" in brackets, you must write your exam paper in Danish.

If you are in doubt about which title you registered for, please see the print of your exam registration from the students' self-service system.

Exam Question 1:

From the rich literature on ARCH models the log-ARCH model for log-returns x_t has been proposed and we shall study a model of this form here. It is given by,

$$x_t = \sigma_t z_t \qquad \log \sigma_t^2 = \mu + \rho \log \left(x_{t-1}^2 \right),$$

with z_t iidN$(0,1)$ for $t = 1, ..., T$ and with x_0 fixed. Moreover, the log-ARCH parameters μ and ρ are real numbers, not necessarily positive.

Question 1.1: Figure 1 shows a simulated sample with (true values) $\mu_0 = 0.1$ and $\rho_0 = 0.4$. Table 1 gives some output from estimation of a (log-)return series.

Comment on Figure 1 (ARCH effects) and Table 1 (in terms of misspecification). Explain how this model differs from the classic ARCH(1) model. Relate it in particular to a classic log SV model.

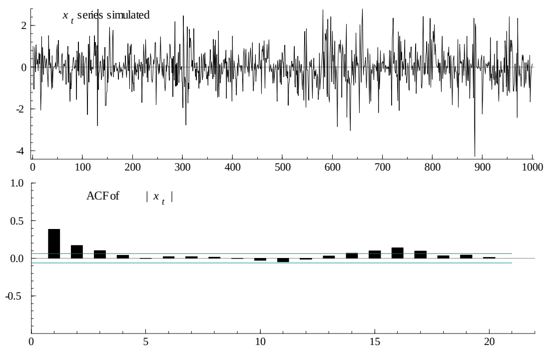

Figure 1: Plot of x_t and ACF of $|x_t|$

Table 1	
Parameter estimate:	$\hat{\rho} = 0.39$ (std.deviation $= 0.054$)
Standarized residuals: $\hat{z}_t := x_t / \hat{\sigma}_t$	
Normality Test for $\hat{z}_t$:	p-value: 0.18
LM test for ARCH in $\hat{z}_t$:	p-value: 0.42

Question 1.2: It is clear that $\log x_t^2$ is a Markov chain. Choose a drift function $\delta(\log x_t^2)$ and show by using this that $\log x_t^2$ is weakly mixing and $E(\log x_t^2)^2 < \infty$ if $\rho^2 < 1$.

Recall that $E \log z_t^2 = 2E \log |z_t| \simeq -1.2$ and $V(\log z_t^2) = 4V(\log |z_t|) = \pi^2/2$, that is, they are well-defined and finite when z_t is N$(0,1)$.

Question 1.3: Next, turn to estimation of the parameter ρ (leaving μ as fixed or known for simplicity). Consider the log-likelihood function $\ell_T(\rho)$ given by,

$$\ell_T(\rho) = -\tfrac{1}{2}\sum_{t=1}^{T}\left(\log\sigma_t^2 + \frac{x_t^2}{\sigma_t^2}\right).$$

Describe briefly how to obtain the MLE of ρ by using *ox* or some other programming language.

Outline 1.4: We know from Theorem III.2 that $\hat{\rho}$ is consistent and asymptotically Gaussian provided regularity conditions hold. A key condition is (A.1) in Theorem III.2 which states that

$$\frac{1}{\sqrt{T}}\,\partial\ell_T(\rho)/\partial\rho\big|_{\rho=\rho_0} \xrightarrow{D} N(0,\Sigma).$$

Show that this holds. Please, be specific about which results you use, verify that they hold and define Σ.

Question 1.5: As an alternative to the MLE we consider OLS. More precisely, based on an AR(1) equation,

$$y_t = \delta + \phi y_{t-1} + \varepsilon_t,$$

it is proposed to estimate ρ and μ by OLS. That is, with $y_t = \log x_t^2$, $\bar{y} = \frac{1}{T}\sum_{t=1}^{T} y_t$, $\bar{y}_{-1} = \frac{1}{T}\sum_{t=1}^{T} y_{t-1}$ define the OLS estimators,

$$\hat{\phi}_{\text{ols}} = \frac{\frac{1}{T}\sum_{t=1}^{T} y_t\left(y_{t-1}-\bar{y}_{-1}\right)}{\frac{1}{T}\sum_{t=1}^{T}\left(y_{t-1}-\bar{y}_{-1}\right)^2}, \quad \hat{\delta}_{\text{ols}} = \bar{y} - \hat{\phi}_{\text{ols}}\bar{y}_{-1}.$$

Use the definition of x_t to argue that $\hat{\phi}_{\text{ols}} \xrightarrow{P} \rho$, that is the OLS estimator is consistent. Would you expect that $\hat{\delta}_{\text{ols}} \xrightarrow{P} \mu$ or something different? Explain your answer.

Exam Question 2:

Question 2.1: Figure 2 shows $\log|x_t|$ where x_t is a log-returns series with $T = 1200$ observations. Furthermore, Table 2.1 shows some output from estimation of an IGARCH(1,1) model for x_t. That is, for the IGARCH the conditional variance is specified as,

$$\sigma_t^2 = \omega + \alpha x_{t-1}^2 + \beta \sigma_{t-1}^2, \qquad \alpha + \beta = 1.$$

From Figure 2 of $\log|x_t|$, would you expect an IGARCH(1,1) to fit the data? Can you conclude something on the basis of the output in Table 2.

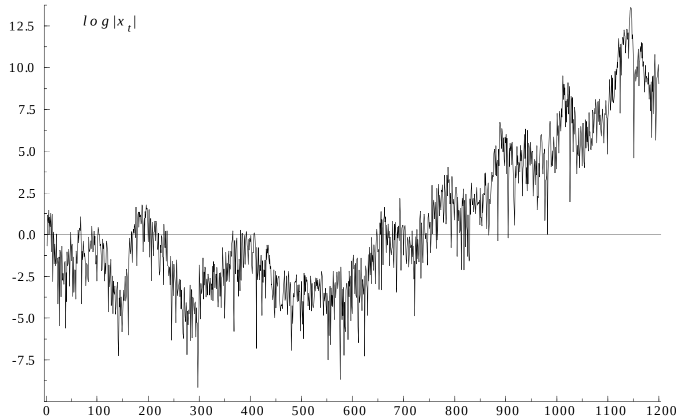

Figure 2: Plot of $\log|x_t|$

Table 2	
Parameter estimates in GARCH(1,1):	$\hat{\alpha} + \hat{\beta} = 1$
Standarized residuals: $\hat{z}_t = x_t/\hat{\sigma}_t$	
Normality Test for $\hat{z}_t$:	p-value: 0.00
LM test for ARCH in $\hat{z}_t$:	p-value: 0.15

Question 2.2: We shall in the following consider the (bivariate) SV model for the x_t series in Figure 2.1 given by,

$$x_t = \sigma_{1t}\sigma_{2t}z_t \quad \text{where}$$
$$\begin{pmatrix} \log\sigma_{1t} \\ \log\sigma_{2t} \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \begin{pmatrix} \log\sigma_{1t-1} \\ \log\sigma_{2t-1} \end{pmatrix} + \begin{pmatrix} \eta_{1t} \\ \eta_{2t} \end{pmatrix},$$

with z_t iidN$(0,1)$ and $\eta_t = (\eta_{1t}, \eta_{2t})'$ is iidN$_2\,(0, \Omega)$, with

$$\Omega = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}.$$

Moreover, z_t and η_t are mutually independent. With

$$\Phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix},$$

find a condition on Φ which ensures that $V_t = (\log\sigma_{1t}, \log\sigma_{2t})'$ is weakly mixing.

Question 2.3: Write the model on a state space form, and use this to suggest a way to estimate the parameters of the model. Is your suggested way of estimation MLE? Explain why or why not this is the case.

Question 2.4: Consider,

$$\mu = (\mu_1, \mu_2)' = (0, 0)'$$
$$\Phi = \begin{pmatrix} 1 & 0 \\ 0 & 0.2 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{pmatrix}.$$

Use this to show that $\log|x_t|$ has the representation,

$$\log|x_t| = \sum_{i=1}^{t} \eta_{1i} + \sum_{i=0}^{\infty} \left(\tfrac{1}{4}\right)^i \eta_{2t-i} + \log|z_t| + \log\sigma_{10}.$$

How does that compare with Figure 2 above?